PROBLEM SET 7 SOLUTIONS.

(1) (3pts) Compute the following definite integrals.
(a) $\int_0^1 x(x^2 + 2)^3 \, dx$
   \textbf{ANSWER:} Letting $u = x^2 + 2$ we have
   \[\frac{1}{2} \int_2^3 u^3 \, du = \frac{65}{8} \approx 8.1.\]
(b) $\int_0^1 xe^x \, dx$
   \textbf{ANSWER:} Integrate by parts with $u = x$ and $dv = e^x \, dx$ to get
   \[\left. (xe^x - e^x) \right|_0^1 = 1.\]
(c) $\int_0^2 \sqrt{4 - x} \, dx$
   \textbf{ANSWER:} Use the substitution $u = 4 - x$. Then $dx = -du$ and the new limits are from $u = 4$ to $u = 2$. We then get
   \[\frac{2}{3} \int_2^4 u^{\frac{3}{2}} \, du = \frac{16 - 4\sqrt{2}}{3} \approx 3.5.\]

(2) (3pts) Find the geometric area of the following functions on the corresponding interval.
(a) $f(x) = 6 - 3x^2$ on $[0, 2]$
   \textbf{ANSWER:} The integral of $f$ is $6x - x^3$. The function $f$ is positive from 0 to $\sqrt{2}$ and negative from $\sqrt{2}$ to 2, so to find the geometric area we need to evaluate
   \[\left. (6x - x^3) \right|_0^{\sqrt{2}} - \left. (6x - x^3) \right|_{\sqrt{2}}^2 = 8\sqrt{2} - 4 \approx 7.3.\]
(b) $f(x) = 3x^2 - 3$ on $[0, 3]$
   \textbf{ANSWER:} The integral of $f$ is $x^3 - 3x$. In this case the function $f$ is negative from 0 to 1 and positive from 1 to 3 so we compute
   \[-\left. (x^3 - 3x) \right|_0^1 + \left. (x^3 - 3x) \right|_1^3 = 22.\]
(c) $f(x) = 9x^2 - 36$ on $[0, 4]$
   \textbf{ANSWER:} The integral of $f$ is $3x^3 - 36x$. Here $f$ is negative from 0 to 2 and positive from 2 to 4 so the area is just
   \[-\left. (3x^3 - 36x) \right|_0^2 + \left. (3x^3 - 36x) \right|_2^4 = 144.\]

(3) (8pts) Compute the following integrals using integration by parts.
(a) \( \int \frac{\ln(x)}{x} \, dx \)

**ANSWER:** You can do this integral by integration by parts (see below), but it is much easier to just substitute \( u = \ln(x) \), because then \( du = \frac{1}{x} \, dx \) and the integral just becomes

\[
\int u \, du = \frac{u^2}{2} + C = \frac{1}{2} (\ln(x))^2 + C.
\]

The integration by parts method is interesting however, because it is an example of an integration by parts that does not yield the answer directly, but rather implicitly: Let \( u = \ln(x) \), \( dv = \frac{1}{x} \, dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = \ln(x) \), so

\[
\int \frac{\ln(x)}{x} \, dx = \ln(x) \ln(x) - \int \frac{\ln(x)}{x} \, dx
\]

Therefore \( 2 \int \frac{\ln(x)}{x} \, dx = (\ln(x))^2 \), and we get the same answer as we do by substitution, even though we never directly computed the integral.

(b) \( \int x^2 e^x \, dx \) (You will have to do the process twice in this example.)

**ANSWER:** Let \( u = x^2 \) and \( dv = e^x \, dx \). The first integration by parts gives us \( x^2 e^x - 2 \int xe^x \, dx \), which must be integrated by parts again, as in 1(b). The final answer is \( (x^2 - 2x + 2)e^x + C \)

(c) \( \int xe^{ax} \, dx \) for a real number \( a \)

**ANSWER:** Let \( u = x \), \( dv = e^{ax} \, dx \).

\[
\int xe^{ax} \, dx = \frac{xe^{ax}}{a} - \frac{1}{a} \int e^{ax} \, dx = \left( \frac{x}{a} - \frac{1}{a^2} \right)e^{ax} + C.
\]

(d) \( \int (\ln(x))^2 \, dx \)

**ANSWER:** Let \( u = (\ln(x))^2 \), \( dv = dx \). Then \( du = 2 \frac{\ln(x)}{x} \, dx \) and \( v = x \). So,

\[
\int (\ln(x))^2 \, dx = x(\ln(x))^2 - 2 \int \ln(x) \, dx.
\]

We can now evaluate \( \int \ln(x) \, dx \) in a similar way, letting \( u = \ln(x) \), \( dv = dx \).
Integration by parts gives \( \int \ln(x) \, dx = x \ln(x) - \int 1 \, dx \). The final answer is:

\[
\int (\ln(x))^2 \, dx = x(\ln(x))^2 - 2x \ln(x) + 2x + C.
\]

(4) (3pts) Find the (geometric) area between the following curves and the \( x \)-axis.

(a) \( f(x) = 27 - 3x^2 \)

**ANSWER:** The indefinite integral is \( 27 - x^3 \), and the curve \( f \) is above the \( x \)-axis between \( x = -3 \) and \( x = 3 \), so

\[
\text{Geometric Area} = (27x - x^3) \bigg|_{-3}^{3} = 108.
\]
(b) \( f(x) = 12 - \frac{3}{4}x^2 \)

**ANSWER:** The indefinite integral is \( 12x - \frac{1}{4}x^3 \), and the curve \( f \) is above the \( x \)-axis between \( x = -4 \) and \( x = 4 \), so,

\[
\text{Geometric Area} = \left. \left( 12x - \frac{1}{4}x^3 \right) \right|_{-4}^{4} = 64.
\]

(c) \( f(x) = -2x - \frac{x^2}{2} \)

**ANSWER:** The indefinite integral is \( -x^2 - \frac{1}{6}x^3 \), and the curve \( f \) is above the \( x \)-axis between \( x = -4 \) and \( x = 0 \), so,

\[
\text{Geometric Area} = \left. \left( -x^2 - \frac{1}{6}x^3 \right) \right|_{-4}^{0} = \frac{16}{3} \approx 5.3.
\]

(5) (3pts) Find the area of the region bounded by the two curves given.

(a) \( f(x) = \cos(x) \) and \( g(x) = \sin(2x) \) on \([0, \frac{\pi}{2}]\) (Hint: \( f(x) = g(x) \) when \( x = \frac{\pi}{6} \)).

**ANSWER:** The function \( \cos(x) \) is greater than \( \sin(2x) \) on the interval \([0, \frac{\pi}{6}]\) and less than \( \sin(2x) \) on \([\frac{\pi}{6}, \frac{\pi}{2}]\), so the area is,

\[
\int_{0}^{\frac{\pi}{6}} (\cos(x) - \sin(2x)) \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin(2x) - \sin(x)) \, dx = \frac{1}{2}.
\]

(b) \( f(x) = x^2 - 4x \) and \( g(x) = 2x \)

**ANSWER:** The two curves intersect where \( x^2 - 4x - 2x = 0 \), i.e. \( x = 0, 6 \), and so the region bounded by the two curves is between \( x = 0 \) and \( x = 6 \). In that interval \( g(x) \geq f(x) \), so the area of the region is

\[
\int_{0}^{6} (2x - x^2 + 4x \, dx) = 36.
\]

(c) \( f(x) = 7 - x^2 \) and \( g(x) = 2 \)

**ANSWER:** The two curves intersect at \( x = \pm \sqrt{5} \). In the interval \([ -\sqrt{5}, \sqrt{5}]\), \( f(x) \geq g(x) \), so the area of the bounded region is,

\[
\int_{-\sqrt{5}}^{\sqrt{5}} (7 - x^2 - 2) \, dx = \frac{20\sqrt{5}}{3} \approx 14.9.
\]